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# ON THE ACCURACY OF ONE-DIMENSIONAL MODELS FOR MULTILAYERED COMPOSITE BEAMS

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**Abstract** The accuracy of 1-D models for composite beams made of linearly elastic orthotropic layers is estimated by means of the Prager Synge hypercircle method. Statically admissible and kinematically admissible stress fields are derived, and asymptotic forms for the error bounds of $O(h/l)$ and $O(h/l)^2$ for $h/l$ approaching zero are obtained for displacement-based laminated beam theories where the axial displacement is represented in terms of a given set of coordinate functions defined over the beam height. The condition of vanishing of relative mean-square error for $h/l \to 0$ is used to derive the constitutive law for the 1-D model. Explicit forms for the error bounds are given for the classical lamination theory, first order shear deformation theory and two higher order theories. Quantitative error bounds are calculated for simply supported multilayered beams under sinusoidal transverse loading. It is shown that, starting from a wise choice of coordinate functions, the accuracy of higher order theories can be almost independent of the beam lamination and up to 150 and 80 times higher than CLT and FSDT, respectively.

## 1. INTRODUCTION

It is widely recognized that a rational foundation of a 1-D theory for the analysis of homogeneous or laminated beams needs to be validated through the estimate of the error connected with the approximate character of the model (Koiter, 1970). Following the classical way, outlined in the Reissner papers (Reissner, 1963), error bounds can be evaluated by constructing statically (S) and kinematically (K) admissible 2-D stress fields as close as possible. Then, the accuracy of a 1-D beam model, when considered as an approximation of the exact 2-D solution, can be estimated by means of the hypercircle method by Prager and Synge (1947).

The effectiveness and comparison of different beam models should be substantiated by checking their capability of describing both the interior solution and the boundary effects. Even though boundary effects are usually very significant for orthotropic beams [see, for instance, Choi and Horgan (1977), Savoia *et al.* (1993), Savoia and Tullini (1994)], with very few exceptions (Rychter, 1987a), error estimates are typically restricted to the interior solution, and in particular to its asymptotic behavior for height-to-length ratio approaching zero.

In spite of the great potentialities of the hypercircle method, rather simple theories are usually considered. For instance, with reference to isotropic beam and plates theories based on the Kirchhoff–Love hypothesis, Koiter (1970) and Nordgren (1971) derived bounds on the relative mean-square error of $O(h/l)$, where $h$ is the beam height and $l$ is a measure of the solution wavelength (Koiter, 1970). Bounds of $O(h^2/l^2)$ have been obtained by Danielson (1971), Simmonds (1971) and Ladevèze (1976). Levinson's beam theory accounting for transverse shear deformations has been considered by Rychter (1987b,c). In Duva and Simmonds (1990, 1991) a relative error of $O(h^N/l^N)$, where $N$ is any positive integer, has been obtained for orthotropic (possibly weak in shear) rectangular beams making use of an asymptotic expansion of the Airy stress function. Corrections due to 2-D end effects have been considered in Duva and Simmonds (1992). Energy bounds for classical and shear-deformable plate theories have been obtained by Nordgren (1972) and Rychter (1987d,e, 1993) for homogeneous anisotropic plates. Ladevèze (1980) showed that, by taking the exact edge conditions into account, the error of classical and Reissner–Mindlin theories cannot be lower than $O(h/l)$, whereas estimates of $O(h^2/l^2)$ must be regarded as interior-domain error only, since the S admissible stress field used does not usually satisfy the prescribed edge conditions (for instance, in the case of a traction-free edge). On the basis of this study, Ladevèze and Pécastaings (1988) proposed an improved version of Reissner's theory (called *Optimal version*) for homogeneous isotropic plates with any boundary conditions which is a second-order approximation of the exact solution. Finally, more accurate displacement fields have been proposed by Rychter (1988a,b, 1992) so obtaining error bounds proportional to the height-to-length ratio cubed.

In this context, to the author's knowledge, the only paper devoted to multilayered plates is due to Van Keulen (1991) who used Danielson's technique to obtain an error bound of $O(h^2/l^2)$ starting from classical lamination theory. Moreover, in Van Keulen (1991) it has been shown that the ratio $h/l$ for which the classical theory can be used safely is much smaller for multilayered (especially fiber-reinforced) plates than for homogeneous isotropic plates under similar conditions.

In none of the referenced papers the error bounds are quantitatively computed, so that they cannot be used to compare the accuracies of different 1-D models. This problem is particularly important for laminated beams. In fact, in spite of the completely different behavior, the same displacement-based models proposed for homogeneous isotropic beams are usually employed, with the mere adoption of appropriate constitutive laws. For instance, classical lamination theory (CLT) (Reissner and Stavsky, 1961) and first order lamination theory (FSDT) (Whitney and Pagano, 1970) represent the straightforward extension of Kirchhoff and Reissner's theories to laminated plates. Moreover, Lo *et al.*'s (1977) higher order model adopts power functions of increasing order over the whole beam height as coordinate functions for displacements. These models are not sufficiently accurate for laminated beams, due to the discontinuity of shear modulus at the layer interfaces requiring discontinuity of shear strain and, consequently, of derivatives of displacements through the beam height (Pagano 1969, Savoia and Reddy, 1992; Savoia *et al.*, 1994).

In this paper, the accuracy of classical and higher order displacement-based models for multilayered beams is discussed. First of all, an energy-consistent derivation of equilibrium equations of 1-D models based on the assumption of transverse inextensibility is performed. Then, with reference to the interior domain problem, an S admissible stress field and a lower accuracy K admissible stress field are derived starting from the 1-D solution. The relative mean-square error is dominated by the difference between the shear stress distributions and is found in the asymptotic form $\hat{C}^{\mathrm{I}} \cdot h/l$ for $h/l \to 0$, where the coefficient $\hat{C}^{\mathrm{I}}$ depends on the beam lamination and the 1-D model adopted.

Subsequently, an improved K admissible solution is constructed, which can be viewed as an extension of Danielson's solution to higher order models for orthotropic multilayered beams; the corresponding relative mean-square error is $\hat{C}^{\mathrm{II}} \cdot (h/l)^2$.

The proposed error bounds apply to all the displacement-based models which represent the axial displacement by means of a linear combination of coordinate functions defined over the beam height and unknown functions defined along the beam axis. The coefficients $C^{\mathrm{I}}$ and $C^{\mathrm{II}}$ can be used to measure the accuracy of the 1-D model adopted as the starting point for deriving the S and the K admissible stress fields. Explicit expressions for these coefficients are obtained for CLT, FSDT and higher order theories, including the Lo *et al.* (1977) model (LHDT) and the refined model proposed by the author (Savoia *et al.*, 1993). The last theory (SHDT) is based on piecewise polynomial $C^0$-continuous coordinate function for the axial displacement, defined over the whole laminate height and selected so as to satisfy the shear stress continuity at the layer interfaces.

The accuracy of 1-D models is quantitatively evaluated in Section 7 with reference to simply-supported multilayered beams under sinusoidal transverse loading. The examples presented show that, unlike the single-layer models based on $C^{\infty}$-continuous coordinate functions (CLT, FSDT, LHDT), the accuracy of SHDT is substantially independent of the beam lamination and the degree of orthotropy of layers. Moreover, the lower-accuracy error bound (related to coefficient $\hat{C}^{\mathrm{I}}$) for SHDT is much narrower than for CLT and LHDT, due to the more accurate representation of shear stresses. On the contrary, improved

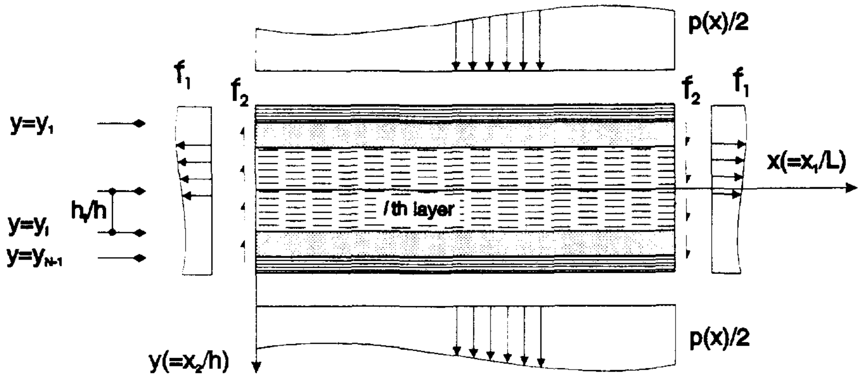



Fig. 1. N-layered orthotropic beam in plane stress.

error bounds for LHDT are only slightly wider than those given by SHDT; in fact, as shown in Section 6, the additional axial displacement corresponding to the improved displacement field for LHDT represents a good approximation of the cross-sectional warping due to shearing stresses.

The numerical examples presented clearly show that evaluation of the asymptotic behavior (for $l/h \to \infty$) only of the error bounds can lead to erroneous conclusions when beams of finite length are considered. For instance for both isotropic and orthotropic laminated beams, the error bound of $O(h^2/l^2)$ for FSDT and for LHDT can be much wider than that of $O(h/l)$ for SHDT if $L/h < 200$ and $L/h < 50$, respectively. This fact represents a strong motivation for the development of 1-D higher order beam theories giving *ab initio* accurate stress distributions.

## 2. THE 2-D ELASTICITY PROBLEM FOR MULTILAYERED BEAMS

A multilayered beam of length $L$ and rectangular cross-section is considered, having unit thickness which is assumed sufficiently small relative to the beam height so that the plane stress hypothesis applies; $x_1$ and $x_2$ axes are chosen in the axial and transverse directions, respectively. Non-dimensional coordinates $x = x_1/L$ and $y = x_2/h$ are introduced (Fig. 1), so that the domain occupied by the beam reduces to $\Omega = [0, 1] \times [-1/2, 1/2]$. The beam is composed of $N$ linearly elastic orthotropic layers, perfectly bonded and symmetrically arranged with respect to the $x$-axis: their thicknesses are denoted by $h_i$ ($i = 1, \ldots N$) and the layer interfaces are located at $y = y_i$ ($i = 1, \ldots, N-1$). The beam is subject to two equally distributed transverse loads $p(x)/2$, acting at the top and bottom faces of the beam ($y = \pm 1/2$) and tractions $f_1, f_2$ at the end sections.

The "interior problem" will be considered here or, equivalently, the prescribed displacement and stress boundary conditions at the beam ends ($x = 0, 1$) are supposed to have the same thickness distribution as the K and the S admissible fields that will be further constructed, respectively.

The governing equations of the 2-D linear elasticity problem are:

(A) *Equilibrium equations*

$$\frac{1}{L}\sigma_{11,x} + \frac{1}{h}\sigma_{12,y} = 0 \quad \frac{1}{L}\sigma_{21,x} + \frac{1}{h}\sigma_{22,y} = 0 \quad \text{on } \Omega; \tag{1a,b}$$

$$[\sigma_{12}] = 0 \quad [\sigma_{22}] = 0 \quad \text{at } y = y_i; \tag{2a,b}$$

$$\sigma_{12} = 0, \quad \sigma_{22} = \pm p(x)/2 \quad \text{at } y = \pm 1/2. \tag{3a,b}$$

(B) *Strain-displacement relations and displacement compatibility at the layer interfaces*

$$\varepsilon_{11} = \frac{1}{L} u_{1,x}, \quad 2\varepsilon_{12} = \frac{1}{h} u_{1,y} + \frac{1}{L} u_{2,x}, \quad \varepsilon_{22} = \frac{1}{h} u_{2,y} \quad \text{on } \Omega; \tag{4a,b,c}$$

$$[\![u_1]\!] = 0, \quad [\![u_2]\!] = 0 \quad \text{at } y = y_i. \tag{5a,b}$$

(C) *Constitutive law for orthotropic layers*

$$\sigma_{11} = C_{1111}\varepsilon_{11} + C_{1122}\varepsilon_{22}, \quad \sigma_{22} = C_{1122}\varepsilon_{11} + C_{2222}\varepsilon_{22}, \quad \sigma_{12} = 2C_{1212}\varepsilon_{12}, \tag{6a,b,c}$$

where

$$\begin{aligned} C_{1111} &= E_1/(1-\nu_{12}\nu_{21}), \quad C_{1122} = \nu_{12}E_2/(1-\nu_{12}\nu_{21}), \\ C_{2222} &= E_2/(1-\nu_{12}\nu_{21}), \quad C_{1212} = G_{12} \end{aligned} \tag{7}$$

and $E_1$, $E_2$ are the Young moduli, $G_{12}$ is the shear modulus and $\nu_{12}$, $\nu_{21}$ are the Poisson ratios. In eqns (1)–(6), comma means partial derivative and the symbol $[\![\cdot]\!]$ stands for jump of the relevant argument; moreover $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$ are the stress and the linear strain tensor and $\mathbf{u}$ is the displacement vector.

## 3. ENERGY-CONSISTENT DERIVATION OF 1-D MODELS BASED ON THE ASSUMPTION OF TRANSVERSE INEXTENSIBILITY

An energy-consistent displacement-based beam model can be obtained by direct substitution of an *a priori* assumed displacement field into the 2-dimensional virtual work statement (Lewinski, 1987). This variational equation states, for the problem at hand:

$$hL\int_{\Omega}\left[\sigma_{11}\frac{1}{L}\tilde{u}_{1,x} + \sigma_{12}\left(\frac{1}{h}\tilde{u}_{1,y} + \frac{1}{L}\tilde{u}_{2,x}\right) + \sigma_{22}\frac{1}{h}\tilde{u}_{2,y}\right]\mathrm{d}\Omega - L\int_0^1 \frac{p(x)}{2}[\tilde{u}_2|_{y=-1/2} + \tilde{u}_2|_{y=1/2}]\,\mathrm{d}x$$

$$-h\left[\int_{-1/2}^{1/2}(f_1\tilde{u}_1 + f_2\tilde{u}_2)\,\mathrm{d}y\right]_{x=0}^{x=1} = 0 \tag{8}$$

to hold for every kinematically admissible displacement field $\tilde{u}_1$, $\tilde{u}_2$. This procedure allows for the direct derivation of the set of equilibrium equations involving the active part of the stress tensor only (Truesdell and Noll, 1965). Most of the 1-D beam models are based on a displacement field which can be written in the form:

$$u_2(x, y) = L\eta(x), \quad u_1(x, y) = hu(x, y), \tag{9}$$

where

$$u(x, y) = -y\varphi(x) + \sum_{s=1}^{n_f}\psi_s(y)\chi_s(x) = -y\varphi(x) + \boldsymbol{\psi}^{\mathrm{T}}(y)\boldsymbol{\chi}(x). \tag{10}$$

Equation (9a) represents the classical kinematic assumption of beam inextensibility in the transverse direction, whereas the axial displacement is represented in eqn (10) by means of a linear combination of coordinate functions $y$ and $\boldsymbol{\psi}$ and unknown functions $\varphi$ and $\boldsymbol{\chi}$. 1-D beam models differ in the number $n_f$ and the type of coordinate functions $\boldsymbol{\psi}$, which are usually selected like the first terms of a complete set of functions, e.g. power functions (Chepiga, 1977; Lo *et al.*, 1977) or Legendre polynomials (Cicala, 1962) over the whole beam height, or $C^0$-continuous functions (Murakami, 1986; Reddy, 1987; Savoia *et al.*, 1993).

Substituting eqn (9) in (8) and performing an integration by part yields:

$$-h\int_{\Omega}\left(\frac{h}{L}\sigma_{11,x}+\sigma_{12,y}\right)\tilde{u}\,\mathrm{d}\Omega - L\int_0^1\left(\frac{h}{L}\int_{-1/2}^{1/2}\sigma_{12,x}\,\mathrm{d}y+p(x)\right)\tilde{\eta}\,\mathrm{d}x$$

$$+h\int_0^1[\sigma_{12}\tilde{u}]_{y=-1/2}^{y=1/2}\,\mathrm{d}x+h\sum_{i=1}^{N-1}\int_0^1[\![\sigma_{12}\tilde{u}]\!]_{y_i}\,\mathrm{d}x+[\ldots]_{x=0}^{x=1}=0 \quad \forall\,\tilde{u}(x,y),\tilde{\eta}(x), \tag{11}$$

where the terms enclosed in the last parentheses are evaluated at the beam end sections. Substituting eqn (10) in (11) and making use of the fundamental lemma of variational calculus yields the following set of equilibrium equations for the interior problem:

$$\frac{h}{L}\frac{\partial}{\partial x}\int_{-1/2}^{1/2}\sigma_{12}\,\mathrm{d}y = -p(x),$$

$$\int_{-1/2}^{1/2}\left(\frac{h}{L}\sigma_{11,x}+\sigma_{12,y}\right)y\,\mathrm{d}y-[\sigma_{12}y]_{y=-1/2}^{y=1/2}-\sum_{i=1}^{N-1}[\![\sigma_{12}y]\!]_{y_i}$$

$$=\frac{h}{L}\int_{-1/2}^{1/2}\sigma_{11,x}y\,\mathrm{d}y-\int_{-1/2}^{1/2}\sigma_{12}\,\mathrm{d}y=0$$

$$\int_{-1/2}^{1/2}\left(\frac{h}{L}\sigma_{11,x}+\sigma_{12,y}\right)\boldsymbol{\psi}\,\mathrm{d}y-[\sigma_{12}\boldsymbol{\psi}]_{y=-1/2}^{y=1/2}-\sum_{i=1}^{N-1}[\![\sigma_{12}\boldsymbol{\psi}]\!]_{y_i}=\mathbf{0}. \tag{12}$$

Equation (12a) does not contain the stress component $\sigma_{22}$, which is a reactive component due to the vanishing of the transverse strain $\varepsilon_{22}$. Hence, the equilibrium condition in the transverse direction can be imposed in a global form only. Moreover, eqns (12b,c) state that the equilibrium in the axial direction is imposed by means of a set of $n_f+1$ (linearly independent) equations. It is worth noting that the first term only in eqns (12b,c) corresponds to the Bollé–Mindlin manner of deriving the equilibrium equations, that is by taking the higher order moments of the equilibrium equations over the beam height (Librescu, 1967; Reissner, 1985). Hence, as pointed out in Lewinski (1987), the Bollé–Mindlin procedure appears to be energy-inconsistent: it would be energy-consistent only if the shear stress $\sigma_{12}$ (written in terms of displacements) satisfies the equilibrium eqns (2a,3a) at the external faces of the beam and at the layer interfaces. Usually these conditions cannot be met, unless functions which *a priori* and individually satisfy equilibrium equations are adopted as coordinate functions.†

As far as the stress–strain relation is concerned, the introduction of an internal constraint in the space of admissible deformations requires a proper definition of constitutive equations involving the active part of the stress tensor only (Truesdell and Noll, 1965; Podio Guidugli, 1989), i.e. the stress components $\sigma_{11}$ and $\sigma_{12}$; hence, the constitutive law for the general layer is written in the form:

$$\sigma_{11}=C^*_{1111}\varepsilon_{11},\quad \sigma_{12}=2C^*_{1212}\varepsilon_{12}. \tag{13}$$

In the following section, the constitutive coefficients $C^*_{1111}$ and $C^*_{1212}$ will be written in terms of the 2-D elastic coefficients of eqns (7) by setting the vanishing of the mean-square error for $h/L$ approaching zero.

Making use of eqns (9, 10), the stress components (13) are rewritten as:

†The Bollé-Mindlin procedure cannot be used even when eqns (2a,3a) are satisfied by prescribing a fixed dependence between the unknown functions $\eta$, $\varphi$ and $\chi$ (see for instance Andreev and Nemirovskii, 1977; Levinson, 1981; Rasskazov *et al.*, 1983). In fact, the equilibrium equations derived from the virtual work principle are substantially different both from the higher order moments of the 2-D equilibrium equations and from eqns (12) (Savoia *et al.*, 1994).

$$\sigma_{11} = \frac{h}{L} C^*_{1111}(-y\varphi' - \boldsymbol{\psi}^{\mathrm{T}}\boldsymbol{\chi}'), \quad \sigma_{12} = C^*_{1212}\left(\eta' - \varphi + \frac{\mathrm{d}\boldsymbol{\psi}^{\mathrm{T}}}{\mathrm{d}y}\boldsymbol{\chi}\right), \tag{14}$$

where prime stands for derivative with respect to $x$. Substitution of eqns (14) in eqns (12) yields the governing equations in terms of the unknown functions $\eta$ and $\boldsymbol{\chi}$:

$$\begin{aligned} &\frac{h}{L} C'_{1212} D_{yy}(\eta' - \varphi)' + \frac{h}{L} C'_{1212} \mathbf{D}^{\mathrm{T}}_{y\psi}\boldsymbol{\chi}' = -p(x), \\ &\left(\frac{h}{L}\right)^2 C'_{1111} I_{yy}\varphi'' + C'_{1212} D_{yy}(\eta' - \varphi) + C'_{1212}\mathbf{D}^{\mathrm{T}}_{y\psi}\boldsymbol{\chi} = 0, \\ &\left(\frac{h}{L}\right)^2 C'_{1111}\mathbf{I}_{\psi\psi}\boldsymbol{\chi}'' - C'_{1212}\mathbf{D}_{\psi\psi}\boldsymbol{\chi} - C'_{1212}\mathbf{D}_{y\psi}(\eta' - \varphi) = \mathbf{0}, \end{aligned} \tag{15}$$

where the following coefficients have been introduced:†

$$\begin{aligned} D_{yy} &= \int_{-1/2}^{1/2} \frac{C^*_{1212}}{C'_{1212}}\,\mathrm{d}y = \sum_{i=1}^{N} \frac{C^{*(i)}_{1212}}{C'_{1212}} h_i, \\ \mathbf{D}_{y\psi} &= \int_{-1/2}^{1/2} \frac{C^*_{1212}}{C'_{1212}}\frac{\mathrm{d}\boldsymbol{\psi}}{\mathrm{d}y}\,\mathrm{d}y = \left[\frac{C^*_{1212}}{C'_{1212}}\boldsymbol{\psi}\right]_{y=-1/2}^{y=1/2} - \sum_{i=1}^{N-1}\left[\!\left[\frac{C^*_{1212}}{C'_{1212}}\boldsymbol{\psi}\right]\!\right]_{y_i}, \\ I_{yy} &= \int_{-1/2}^{1/2} \frac{C^*_{1111}}{C'_{1111}} y^2\,\mathrm{d}y, \quad \mathbf{I}_{\psi\psi} = \int_{-1/2}^{1/2} \frac{C^*_{1111}}{C'_{1111}}\boldsymbol{\psi}\boldsymbol{\psi}^{\mathrm{T}}\,\mathrm{d}y, \quad \mathbf{D}_{\psi\psi} = \int_{-1/2}^{1/2} \frac{C^*_{1212}}{C'_{1212}}\frac{\mathrm{d}\boldsymbol{\psi}}{\mathrm{d}y}\frac{\mathrm{d}\boldsymbol{\psi}^{\mathrm{T}}}{\mathrm{d}y}\,\mathrm{d}y \end{aligned} \tag{16}$$

and $C'_{1111}, C'_{1212}$ are arbitrary reference elastic moduli. Equations (15) have been derived by imposing the vanishing of the following coupling coefficients:

$$\mathbf{I}_{y\psi} = \int_{-1/2}^{1/2} \frac{C^*_{1111}}{C'_{1111}} y\boldsymbol{\psi}\,\mathrm{d}y = \mathbf{0}. \tag{17}$$

Of course, starting from any set of coordinate functions $\bar{\boldsymbol{\psi}}(y)$, a set of functions $\boldsymbol{\psi}(y)$ satisfying eqn (17) can be easily obtained by making use of a Gram–Schmidt orthogonalization process.

Equations (15) can be specialized with reference to many higher order beam models, since only the calculation of the coefficients reported in eqns (16) is required. For instance, if no additional terms $\boldsymbol{\psi}(y)\,\boldsymbol{\chi}(x)$ are used for the axial displacement (10), equilibrium eqns (15) reduce to the "kinematic version" (that is to the accuracy of the shear correction factor) of the First Order (Reissner Mindlin) laminated beam theory:

$$\frac{h}{L} C'_{1212} D_{yy}(\eta' - \varphi)' = -p(x), \quad \left(\frac{h}{L}\right)^2 C'_{1111} I_{yy}\varphi'' + C'_{1212} D_{yy}(\eta' - \varphi) = 0. \tag{18}$$

## 4. ERROR ESTIMATE FOR 1-D BEAM MODELS

The accuracy of 1-D models when considered as approximations to the original 2-D problem can be estimated by means of the hypersphere theorem of Prager and Synge

†The following property has been used to derive eqn (16e):

$$\int_{-1/2}^{1/2} C^*_{1212}\boldsymbol{\psi}\frac{\mathrm{d}^2\boldsymbol{\psi}^{\mathrm{T}}}{\mathrm{d}y^2}\,\mathrm{d}y = -\int_{-1/2}^{1/2} C^*_{1212}\frac{\mathrm{d}\boldsymbol{\psi}}{\mathrm{d}y}\frac{\mathrm{d}\boldsymbol{\psi}^{\mathrm{T}}}{\mathrm{d}y}\,\mathrm{d}y + \left[C^*_{1212}\boldsymbol{\psi}\frac{\mathrm{d}\boldsymbol{\psi}^{\mathrm{T}}}{\mathrm{d}y}\right]_{y=-1/2}^{y=1/2} - \sum_{i=1}^{N-1}\left[\!\left[C^*_{1212}\boldsymbol{\psi}\frac{\mathrm{d}\boldsymbol{\psi}^{\mathrm{T}}}{\mathrm{d}y}\right]\!\right]_{y_i}$$

(1947). It is particularly meaningful to obtain error bounds for the energy error where the dependence on the height-to-length ratio of the beam is made explicit.

Denoting by $\bar{\boldsymbol{\sigma}}$ and $\bar{\bar{\boldsymbol{\sigma}}}$ two S and K admissible 2-D stress fields derived from the 1-D solution, the Prager–Synge hypersphere theorem states that the exact solution $\boldsymbol{\sigma}$ is bounded by the inequality:

$$\frac{\|\bar{\boldsymbol{\sigma}}-\boldsymbol{\sigma}\|}{\|\bar{\boldsymbol{\sigma}}\|} \leqslant \bar{e}, \tag{19}$$

where $\bar{e} = \|\bar{\boldsymbol{\sigma}}-\bar{\bar{\boldsymbol{\sigma}}}\|/\|\bar{\boldsymbol{\sigma}}\|$ is the computable relative error. Equation (19) provides for an error bound when the S-admissible stress field $\bar{\boldsymbol{\sigma}}$ is used as approximation to the exact solution $\boldsymbol{\sigma}$. Of course, the actual relative error $e = \|\bar{\boldsymbol{\sigma}}-\bar{\bar{\boldsymbol{\sigma}}}\|/\|\boldsymbol{\sigma}\|$ can be computed only when the exact solution $\boldsymbol{\sigma}$ is known; nevertheless, as shown in Van Keulen (1991) the computable relative error $\bar{e}$ has the same asymptotic form as the actual relative error. The symbol $\|\cdot\|$ denotes a mean-square norm for the stress tensor, which is based on the (positive-definite) complementary elastic energy functional:

$$\|\boldsymbol{\sigma}\|^2 = Lh\langle S_{1111}\sigma_{11}^2 + S_{2222}\sigma_{22}^2 + 2S_{1122}\sigma_{11}\sigma_{22} + S_{1212}\sigma_{12}^2\rangle_\Omega, \tag{20}$$

where $\mathbf{S} = \mathbf{C}^{-1}$ is the compliance constitutive tensor and the operator $\langle\cdot\rangle_\Omega$ is defined as:

$$\langle f(x,y)\rangle_\Omega = \sqrt{\int_{-1/2}^{1/2}\int_0^1 f(x,y)\,\mathrm{d}x\,\mathrm{d}y}. \tag{21}$$

In the following, when the function $f(x,y)$ can be expressed in terms of separation of variables as $f(x,y) = f_l(x)f_h(y)$, the following notation is used:

$$\langle f(x,y)\rangle_\Omega = \langle f_l(x)\rangle_l \cdot \langle f_h(y)\rangle_h, \tag{22}$$

where

$$\langle f_l(x)\rangle_l = \sqrt{\int_0^1 f_l(x)\,\mathrm{d}x}, \quad \langle f_h(y)\rangle_h = \sqrt{\int_{-1/2}^{1/2} f_h(y)\,\mathrm{d}y}. \tag{23}$$

4.1. *Statically admissible 2-D stress distribution*

A stress field $\bar{\boldsymbol{\sigma}} = \{\bar{\sigma}_{11}, \bar{\sigma}_{12}, \bar{\sigma}_{22}\}$ is statically admissible if it satisfies the 2-D equilibrium eqns (1–3). Starting from the stresses obtained from a 1-D beam model (denoted by $\sigma^*_{11}, \sigma^*_{12}$),† an S admissible stress field can be derived by assuming $\bar{\sigma}_{11}$ coincident with $\sigma^*_{11}$ and obtaining shear and transverse normal stresses by performing the integration of equilibrium eqns (1a,b) over the beam height, making use of the stress continuity (2a,b) at the layer interfaces and of stress balance (3a,b) at $y = \pm 1/2$:

$$\bar{\sigma}_{11} = \sigma^*_{11} = C^*_{1111}\frac{h}{L}u_{,x}, \quad \bar{\sigma}_{12} = \left(\frac{h}{L}\right)^2 f_{12}, \quad \bar{\sigma}_{22} = \left(\frac{h}{L}\right)^3 f_{22}, \tag{24}$$

where

$$f_{12}(x,y) = -\int_{-1/2}^{y} C^*_{1111}u_{,xx}\,\mathrm{d}\hat{y}, \quad f_{22}(x,y) = -\int_0^{y} f_{12,x}\,\mathrm{d}\hat{y} \tag{25}$$

and $u(x,y)$ is the dimensionless axial displacement defined in eqn (10). It is easy to verify

†Displacements, stresses and strains corresponding to the 1-D solution are marked with an asterisk.

that, by virtue of the overall equilibrium in the transverse direction expressed by eqn (12a), the stress field (24) satisfies the stress balance (3a,b) at $y = 1/2$.

### 4.2. *Lower accuracy K admissible 2-D solution*

With any given 1-D displacement field $u_1^*, u_2^*$, a $K$-admissible 2-D displacement field $u_1, u_2$ and the related stress field $\bar{\bar{\sigma}} = \{\bar{\bar{\sigma}}_{11}, \bar{\bar{\sigma}}_{12}, \bar{\bar{\sigma}}_{22}\}$ can be associated, such that:

$$u_1(x, y) = u_1^*(x, y) = hu(x, y), \tag{26a}$$

$$u_2(x, 0) = u_2^*(x, 0) = L\eta(x), \tag{26b}$$

$$\bar{\bar{\sigma}}_{22}(x, y) = 0. \tag{26c}$$

Making use of eqns (6b) and (26a,c) the transverse extension for each layer is obtained in the form:

$$\varepsilon_{22} = -\frac{C_{1122}}{C_{2222}}\varepsilon_{11}^* = -\frac{C_{1122}}{C_{2222}}\frac{h}{L}u_{,x}. \tag{27}$$

Therefore, by integrating eqn (4c) making use of interface compatibility condition (5b), the following expression for the transverse displacement is obtained:

$$u_2 = L\left[\eta - \left(\frac{h}{L}\right)^2 \Delta u_2\right], \quad \Delta u_2(x, y) = \int_0^y \frac{C_{1122}}{C_{2222}} u_{,x}\,\mathrm{d}\hat{y}. \tag{28}$$

The second term in eqn (28a) represents the transverse deformation of the beam related to the Poisson effect. The addition of this term is essential to obtain a bound on the error approaching zero when $h/L \to 0$.

From eqns (4), (6), (26a) and (28) the following K admissible stress field is derived:

$$\bar{\bar{\sigma}}_{11} = C_{1111}\left(1 - \frac{C_{1122}^2}{C_{1111}C_{2222}}\right)\frac{h}{L}u_{,x}, \quad \bar{\bar{\sigma}}_{12} = \frac{C_{1212}}{C_{1212}^*}\sigma_{12}^* - C_{1212}\left(\frac{h}{L}\right)^2 \Delta u_{2,x}, \quad \bar{\bar{\sigma}}_{22} = 0. \tag{29}$$

It is worth noting that the addition of $\Delta u_2$ in eqn (28a) gives rise to a "spurious" contribution for the shear stress $\bar{\bar{\sigma}}_{12}$ in eqn (29b). In fact, as will be shown in Section 6, the shear stress $\bar{\bar{\sigma}}_{12}$ does not represent an improvement over that given by the 1-D model and, in some cases, can be even worse than the original 1-D shear stress.

Following Koiter (1970), the consistency of a 1-D model requires the possibility of constructing 2-D S and K admissible stress fields whose relative error approaches zero when $h/L \to 0$. Equations (24) state that $\bar{\sigma}_{11}$ is the largest component of the stress tensor, so that the consistency requirement can be fulfilled only if the S and K admissible normal stresses $\bar{\sigma}_{11}$ and $\bar{\bar{\sigma}}_{11}$ of eqns (24a) and (29a) coincide. This condition yields the following expressions for the Young modulus of the 1-D constitutive law (13):

$$C_{1111}^* = C_{1111}\left(1 - \frac{C_{1122}^2}{C_{1111}C_{2222}}\right), \tag{30}$$

which represents the direct extension to orthotropic materials of the classical reduced stiffness coefficient. For a beam under plane stress (see eqns 7), eqn (30) reduces to $C_{1111}^* = E_1$. As for the shear modulus, the most natural choice is to assume $C_{1212}^* = C_{1212}$, so that the first term at the RHS of eqn (29b) reduces to the 1-D shear stress $\sigma_{12}^*$.

Due to the relative magnitude of the stress components given by eqns (24), since $\bar{\sigma}_{11}$ has been set equal to $\bar{\bar{\sigma}}_{11}$, the computable relative error bound $\bar{e}$ is dominated by the difference of shear stresses reported in eqns (24b) and (29b) and can be given the following asymptotic form for $h/L \to 0$:

$$\bar{e}^{\mathrm{I}} \approx C^{\mathrm{I}} \frac{h}{L} + O\left(\frac{h}{L}\right)^2. \tag{31}$$

where the coefficient $C^{\mathrm{I}}$, given by:

$$C^{\mathrm{I}} = \frac{\langle S_{1212}(f_{12} - (L/h)^2 \sigma^*_{12} + C_{1212} \Delta u_{2,x})^2 \rangle_\Omega}{\langle S_{1111} C^{*2}_{1111} u^2_{,x} \rangle_\Omega} \tag{32}$$

depends on the beam lamination and loading condition as well as on the model used for the 1-D solution. This coefficient, which will be named "I-order accuracy coefficient" in the following, can be used to estimate the accuracy of the 1-D model.

The displacement field $u_1$, $u_2$ of eqns (26a) and (28) represents an extension to shear-deformable multilayered beams of the modified Kirchhoff-type displacement field obtained by Koiter (1970) in the framework of the classical theory of isotropic shells. In particular, eqn (26b) states that the transverse displacement $u_2^* = L\eta(x)$ given by the 1-D model has been assumed as the midplane displacement of the beam when considered as a 2-D body. Of course, this is not the only possible choice. For instance, Reissner (1944) considered $\eta(x)$ as the transverse displacement weighted over the beam height by the function 1-$(2y)^2$. With reference to classical plate theory, Rychter (1993) derived an infinite family of possible weight functions that can be adopted to this purpose. It is to be noted that for the problem discussed here (a symmetrically laminated beam subject to two identical transverse loads at the top and bottom faces) the transverse displacement $u_2$ in eqn (28) varies along $y$ according to an even law, so that all the proposed definitions coincide.

### 4.3. *Higher accuracy K admissible 2-D solution*

By inspection of eqns (31) and (32) it is evident that in order to improve the error bound it is necessary to make the shear stress $\bar{\bar{\sigma}}_{12}$ close to $\bar{\sigma}_{12}$. To this purpose, a second K admissible solution is proposed by introducing an additional term for the axial displacement so as to make the new shear stress equal to the S admissible shear stress $\bar{\sigma}_{12}$ of eqn (24b), i.e.

$$u_1 = h\left[u + \left(\frac{h}{L}\right)^2 \Delta u_1\right]. \tag{33}$$

where

$$\Delta u_1 = \int_0^{\hat{y}} S_{1212}\left[f_{12} - \left(\frac{L}{h}\right)^2 \sigma^*_{12}\right] \mathrm{d}\hat{y} + \int_0^{\hat{y}} \Delta u_{2,x} \, \mathrm{d}\hat{y}. \tag{34}$$

The first integral in eqn (34) vanishes if the shear and the normal stress distributions $\sigma^*_{11}$, $\sigma^*_{12}$ given by the 1-D model satisfy the equilibrium eqns (1a), (2a) and (3a). The second term is required to remove the "spurious" term appearing in the shear stress (29b). The K admissible stress field obtained from the improved displacement field (28), (33) is:

$$\bar{\bar{\sigma}}_{11} = \bar{\sigma}_{11} + C_{1111}\left(\frac{h}{L}\right)^3 \Delta u_{1,x}, \quad \bar{\bar{\sigma}}_{12} = \bar{\sigma}_{12}, \quad \bar{\bar{\sigma}}_{22} = C_{1122}\left(\frac{h}{L}\right)^3 \Delta u_{1,x}. \tag{35}$$

Hence, the difference between the S and the K admissible stress fields $\bar{\boldsymbol{\sigma}}$ and $\bar{\bar{\boldsymbol{\sigma}}}$ is now related to terms of order $(h/L)^2\,O(\bar{\sigma})$. Making use of eqns (24) and (35), an improved (II-order) bound for the computable relative error is obtained in the form:

$$\bar{e}^{\mathrm{II}} \approx C^{\mathrm{II}}\left(\frac{h}{L}\right)^2, \tag{36}$$

where the "II order accuracy coefficient" $C^{\mathrm{II}}$ is:

$$C^{\mathrm{II}} = \frac{\langle S_{1111} C_{1111}^2 \Delta u_{1,x}^2 + S_{2222}(f_{22} - C_{1122}\Delta u_{1,x})^2 + 2S_{1122}C_{1111}\Delta u_{1,x}(f_{22} - C_{1122}\Delta u_{1,x})\rangle_\Omega}{\langle S_{1111} C_{1111}^{*2} u_{,x}^2\rangle_\Omega}. \tag{37}$$

The procedure used to obtain the improved displacement field is reminiscent of that proposed by Danielson (1971) in the context of classical theory of isotropic shells. The derivation given here can be used to obtain error bounds for any 1-D laminated beam model based on the assumption of transverse inextensibility.

## 5. ERROR ESTIMATES FOR SOME 1-D BEAM MODELS

For most of the 1-D displacement-based theories, the relative mean-square error $\bar{e}$ can be given an explicit form by deriving S and K admissible stress fields from eqns (24), (29) and (35). In this Section, error bounds are obtained for CLT, FSDT and the higher order theories based on the displacement field (9), (10) (Table 1).

### 5.1. *Classical lamination theory (CLT)*

Classical lamination theory, which represents the extension of the classical Euler–Bernoulli model to multilayered beams, is based on the following displacement field and equilibrium equations:

$$u_1^*(x,y) = -hy\eta'(x), \quad u_2^*(x,y) = L\eta(x) \tag{38}$$

$$M(x) = -\frac{h^3}{L} C_{1111}^r I_{yy}\eta''(x), \tag{39}$$

where $M(x) = h^2 \int_{-1/2}^{1/2} \sigma_{11} y\,\mathrm{d}y$ is the bending moment. Starting from the axial stress distribution $\sigma_{11}^*$ given by eqns (13a) and (38a):

$$\sigma_{11}^* = -\frac{h}{L} C_{1111}^* y\eta''(x) \tag{40}$$

an S admissible stress field can be obtained in the form of eqns (24), where:

Table 1. Axial displacement, higher order coordinate functions and equilibrium equations for the classical lamination theory (CLT), first order shear deformation theory (FSDT) and the higher order models by Lo *et al.* (1977) (LHDT) and Savoia *et al.* (1993) (SHDT)

| Model | $u_1(x,y)/h$ | $\psi_n(y)$ | Equilibrium eqns |
|---|---|---|---|
| CLT | $-y\eta'(x)$ | — | eqn (39) |
| FSDT | $-y\varphi(x)$ | | eqns (18) |
| LHDT | $-y\varphi(x)+\boldsymbol{\psi}^{\mathrm{T}}(y)\boldsymbol{\chi}(x)$ | $y^n$ | eqns (15) |
| SHDT | $-y\varphi(x)+\boldsymbol{\psi}^{\mathrm{T}}(y)\boldsymbol{\chi}(x)$ | $C^0$ continuous functions (eqns 62) and (63) | eqns (15) |

$$f_{12}(x,y) = \eta'''(x)\bar{s}_{12}(y), \quad f_{22}(x,y) = -\eta''''(x)\bar{s}_{22}(y) \tag{41}$$

and

$$\bar{s}_{12}(y) = \int_{-1/2}^{y} C^{*}_{1111}\hat{y}\,\mathrm{d}\hat{y}, \quad \bar{s}_{22}(y) = \int_{0}^{y} \bar{s}_{12}(\hat{y})\,\mathrm{d}\hat{y}. \tag{42}$$

As for the K admissible solution, the additional transverse displacement and the related shear stress are given by [see eqns (28) and (29b)]:

$$\Delta u_2 = -\eta''(x)n(y), \quad \bar{\bar{\sigma}}_{12} = \left(\frac{h}{L}\right)^2 \eta'''(x)\bar{\bar{s}}_{12}(y), \tag{43}$$

where

$$n(y) = \int_{0}^{y} \frac{C_{1122}}{C_{2222}}\hat{y}\,\mathrm{d}\hat{y}, \quad \bar{\bar{s}}_{12}(y) = C_{1212}n(y). \tag{44}$$

Hence, making use of eqns (24), (40), (41) and (43b), the asymptotic form of error bound of eqn (31) is obtained, where the corresponding accuracy coefficient is:

$$C^{\mathrm{I}}_{(\mathrm{CLT})} = \frac{\langle \eta'''^2 \rangle_l}{\langle \eta''^2 \rangle_l} \frac{\langle S_{1212}[\bar{s}_{12}(y) - \bar{\bar{s}}_{12}(y)]^2 \rangle_h}{\langle S_{1111} C^{*2}_{1111} y^2 \rangle_h}. \tag{45}$$

Making use of eqn (39), the first term at the R.H.S. of eqn (45) can be set equal to $L/l_1$, where $l_1$ is the solution wavelength which depends on the only loading condition through the axial variation of bending moment $M(x)$ and shear resultant $T(x) = M^{[1]}(x)$:†

$$l_1 = \frac{\langle M^2 \rangle_L}{\langle M^{[1]2} \rangle_l}. \tag{46}$$

Correspondingly, the lower order error bound can be rewritten as a function of the height-to-wavelength ratio:

$$\bar{e}^{\mathrm{I}} \approx \hat{C}^{\mathrm{I}}_{(\mathrm{CLT})}\frac{h}{l_1}, \quad \hat{C}^{\mathrm{I}}_{(\mathrm{CLT})} = \frac{\langle S_{1212}[\bar{s}_{12}(y) - \bar{\bar{s}}_{12}(y)]^2 \rangle_h}{\langle S_{1111} C^{*2}_{1111} y^2 \rangle_h}. \tag{47}$$

As for the improved error bound, CLT gives $\sigma^{*}_{12} = 0$, and the additional axial displacement $\Delta u_1$ defined in eqn (34) reduces to:

$$\Delta u_1 = \eta'''(x)r(y), \quad r(y) = \int_{0}^{y} (S_{1212}\bar{s}_{12}(\hat{y}) - n(\hat{y}))\,\mathrm{d}\hat{y}. \tag{48}$$

Finally, the II-order error bound can be expressed as:

$$\bar{e}^{\mathrm{II}} \approx \hat{C}^{\mathrm{II}}_{(\mathrm{CLT})}\left(\frac{h}{l_2}\right)^2, \tag{49}$$

where

†Bracket stands for derivative with respect to $x$, so that $M^{[1]} = M' \cdot L$.

$$l_2 = \frac{\langle M^2 \rangle_L}{\langle M'^2 \rangle_L} \tag{50}$$

and

$$\hat{C}^{\mathrm{II}}_{(\mathrm{CLT})} = \frac{\langle S_{1111} C^2_{1111} v^2 + S_{2222}[\bar{s}_{22} + C_{1122} v]^2 + 2S_{1122} C_{1111} v[\bar{s}_{22} + C_{1122} v] \rangle_h}{\langle S_{1111} C^{*2}_{1111} v^2 \rangle_h}.$$

It is worth noting that coefficients $\hat{C}^{\mathrm{I}}_{(\mathrm{CLT})}$ and $\hat{C}^{\mathrm{II}}_{(\mathrm{CLT})}$ defined in eqns (47b) and (50b) depend on the beam lamination only, whereas the dependence on the loading condition is restricted to the definition of solution wavelengths $l_1$ and $l_2$.

### 5.2. *Higher order shear deformation theories*

The procedure described in Section 4 can be used to obtain error bounds for any higher order shear deformation theory (HSDT) governed by equilibrium eqns (15). The analytical details are reported in the Appendix.

The shear stress obtained from the 1-D model can be written in the form:

$$\sigma^*_{12} = \left(\frac{h}{L}\right)^2 [s_{12v}(y)\varphi'' - \mathbf{s}^{\mathrm{T}}_{12\psi}(y)\boldsymbol{\chi}''], \tag{51}$$

where the functions:

$$s_{12v}(y) = C_{1212}\left(-g_v + \frac{\mathrm{d}\boldsymbol{\psi}^{\mathrm{T}}}{\mathrm{d}y}\mathbf{e}_v\right), \quad \mathbf{s}^{\mathrm{T}}_{12\psi}(y) = C_{1212}\left(\mathbf{g}^{\mathrm{T}}_{\psi} - \frac{\mathrm{d}\boldsymbol{\psi}^{\mathrm{T}}}{\mathrm{d}y}\mathbf{E}_{\psi}\right) \tag{52}$$

represent the variations over the beam height of the contributions to the shear stress $\sigma^*_{12}$ related to $\varphi''$ and $\boldsymbol{\chi}''$, respectively, and $g_v$, $\mathbf{e}_v$, $\mathbf{g}_{\psi}$, $\mathbf{E}_{\psi}$ are sets of coefficients which depend on the beam lamination and the set of coordinate functions adopted. Moreover, the S admissible shear and transverse normal stress, the additional transverse displacement and the K admissible shear stress are obtained in the form:

$$\bar{\sigma}_{12} = \left(\frac{h}{L}\right)^2 f_{12} = \left(\frac{h}{L}\right)^2 [\bar{s}_{12v}(y)\varphi'' - \bar{\mathbf{s}}^{\mathrm{T}}_{12\psi}(y)\boldsymbol{\chi}''],$$

$$\bar{\sigma}_{22} = \left(\frac{h}{L}\right)^3 f_{22} = \left(\frac{h}{L}\right)^3 [-\bar{s}_{22v}(y)\varphi''' + \bar{\mathbf{s}}^{\mathrm{T}}_{22\psi}(y)\boldsymbol{\chi}'''],$$

$$\Delta u_2 = -n_v(y)\varphi' + \mathbf{n}^{\mathrm{T}}_{\psi}(y)\boldsymbol{\chi}',$$

$$\bar{\bar{\sigma}}_{12} = \left(\frac{h}{L}\right)^2 [\bar{\bar{s}}_{12v}(y)\varphi'' - \bar{\bar{\mathbf{s}}}^{\mathrm{T}}_{12\psi}(y)\boldsymbol{\chi}''], \tag{53}$$

where $\bar{s}_{12v}(y)$, $\bar{s}_{22v}(y)$, $n_v(y)$ coincide with $\bar{s}_{12}(y)$, $\bar{s}_{22}(y)$, $n(y)$ defined in eqns (42) and (44a), and:

$$\bar{\mathbf{s}}_{12\psi}(y) = \int_{-1/2}^{y} C^*_{1111}\boldsymbol{\psi}\,\mathrm{d}\hat{y}, \quad \bar{\mathbf{s}}_{22\psi}(y) = \int_{-1/2}^{y} \bar{\mathbf{s}}_{12\psi}\,\mathrm{d}\hat{y}, \quad \mathbf{n}_{\psi}(y) = \int_0^y \frac{C_{1122}}{C_{2222}}\boldsymbol{\psi}\,\mathrm{d}\hat{y},$$

$$\bar{\bar{s}}_{12v}(y) = -s_{12v}(y) + C_{1212}n_v(y), \quad \bar{\bar{\mathbf{s}}}_{12\psi}(y) = -\mathbf{s}_{12\psi}(y) + C_{1212}\mathbf{n}_{\psi}(y). \tag{54}$$

Finally, making use of eqns (14a), (51) and (53a,c), the asymptotic error estimate (31) is recovered, with:

$$C^{\mathrm{I}}_{(\mathrm{HSDT})} = \frac{\langle S_{1212}[(\bar{s}_{12v} - s_{12v})\varphi'' - (\bar{\mathbf{s}}^{\mathrm{T}}_{12\psi} - \bar{\bar{\mathbf{s}}}^{\mathrm{T}}_{12\psi})\boldsymbol{\chi}'']^2 \rangle_\Omega}{\langle S_{1111} C^{*2}_{11} y^2 \varphi'^2 \rangle_\Omega}. \tag{55}$$

where the condition $\boldsymbol{\chi} \sim (h/L)^2 \varphi$ is used to remove the normal stress related to the higher order contribution $\boldsymbol{\psi}^{\mathrm{T}} \boldsymbol{\chi}'$ at the denominator [see eqns (A3) in the Appendix]. Finally, making use of eqns (A5), the lower order error bound can be written as:

$$\bar{e}^{\mathrm{I}} \approx \hat{C}^{\mathrm{I}}_{(\mathrm{HSDT})} \frac{h}{l_1}, \quad \hat{C}^{\mathrm{I}}_{(\mathrm{HSDT})} = \frac{\langle S_{1212}[(\bar{s}_{12v} - s_{12v}) - (\bar{\mathbf{s}}^{\mathrm{T}}_{12\psi} - \bar{\bar{\mathbf{s}}}^{\mathrm{T}}_{12\psi})\mathbf{c}_\psi / c_v]^2 \rangle_h}{\langle S_{1111} C^{*2}_{11} y^2 \rangle_h}. \tag{56}$$

where $l_1$ is the solution wavelength defined in eqn (46). Equation (56) gives error bounds for 1-D models based on displacement field (9), (10). For instance, FSDT retains the cross-sectional rotation $\varphi(x)$ only in the axial displacement, and the second term in eqn (56) disappears.

As for the improved K admissible solution, the (dimensionless) additional axial displacement is:

$$\Delta u_1 = v_v(y)\varphi''' + \mathbf{v}^{\mathrm{T}}_\psi(y)\boldsymbol{\chi}''', \tag{57}$$

where

$$v_v(y) = \int_0^y [S_{1212}(\bar{s}_{12v} - s_{12v}) - n_v]\,\mathrm{d}\hat{y}, \quad \mathbf{v}_\psi(y) = \int_0^y [S_{1212}(\bar{\mathbf{s}}_{12\psi} - \mathbf{s}_{12\psi}) - \mathbf{n}_\psi]\,\mathrm{d}\hat{y} \tag{58}$$

and the improved error estimate can be written in the form of eqn (36), where

$$\bar{e}^{\mathrm{II}} \approx \hat{C}^{\mathrm{II}}_{(\mathrm{HSDT})} \left(\frac{h}{l_2}\right)^2, \quad \hat{C}^{\mathrm{II}}_{(\mathrm{HSDT})} = \frac{\langle F_v(y) + \mathbf{F}^{\mathrm{T}}_m(y)\mathbf{c}_\psi / c_v + \mathbf{c}^{\mathrm{T}}_\psi \mathbf{F}_\psi(y)\mathbf{c}_\psi / c_v^2 \rangle_h}{\langle S_{1111} C^{*2}_{11} y^2 \rangle_h} \tag{59}$$

and $F_v$, $\mathbf{F}_m$, $\mathbf{F}_\psi$ are sets of functions defined over the beam height, which are not here reported for the sake of brevity. For instance, for FSDT eqn (59b) reduces to:

$$\hat{C}^{\mathrm{II}}_{(\mathrm{FSDT})} = \frac{\langle S_{1111} C^{2}_{1111} v_v^2 + S_{2222}[\bar{s}_{22v} + C_{1122} v_v]^2 + 2S_{1122} C_{1111} v_v [\bar{s}_{22v} + C_{1122} v_v] \rangle_h}{\langle S_{1111} C^{*2}_{11} y^2 \rangle_h}. \tag{60}$$

Coefficients $\hat{C}^{\mathrm{I}}_{(\mathrm{HSDT})}$ and $\hat{C}^{\mathrm{II}}_{(\mathrm{HSDT})}$ reported in eqns (56b) and (59b) depend on the beam lamination as well as on the set of coordinate functions. Hence, the error bounds defined in eqns (56a) and (59a) are useful to estimate the accuracy of stress fields derived from different 1-D models. In the numerical examples (Section 7) two higher order theories will be considered, presented in Lo *et al.* (1977) and Savoia *et al.* (1993).

The Lo–Christensen–Wu higher-order theory (LHDT) adopts a set of power functions of increasing (odd) order defined over the whole beam height as coordinate functions. The procedure developed in the present paper applies to it if a linear combination of power functions is adopted, satisfying the orthogonality condition (17). The main drawback of this model is the fact that, even though a complete set of coordinate functions is adopted, it is no able to yield accurate transverse shear stresses [see, for instance, Savoia *et al.* (1994)]. In fact, using $C^0$-continuous coordinate functions, the shear stress discontinuity at the layer interfaces is equal to the value of the shear strain multiplied by the jump of the shear modulus [see eqn (14b)]:

$$[\![\sigma_{12}]\!] = [\![C_{1212}]\!]2\varepsilon_{12} = [\![C_{1212}]\!]\left(\eta' - \varphi + \frac{d\boldsymbol{\psi}^{T}}{dy}\bigg|_{y_i}\boldsymbol{\chi}\right) \tag{61}$$

and cannot be reduced even if a large number of coordinate functions is adopted. Hence, the I-order accuracy error bound, being dominated by the difference of shear stresses, is expected to be rather wide for laminated beams.

The higher order theory proposed by Savoia *et al.* (SHDT) overcomes this drawback by adopting a set of coordinate $C^0$-continuous functions defined as:

$$\psi_n(y) = \bar{\psi}_n(y) - \frac{I_{y\bar{\psi}_n}}{I_{yy}}y - \sum_{i=1}^{n-1}\frac{I_{\psi_i\bar{\psi}_n}}{I_{\psi_i\psi_i}}\psi_i(y), \quad n = 1,\ldots,n_f \tag{62}$$

where functions $\bar{\psi}_n(y)$ are obtained from the following recursive formula:

$$\frac{d\bar{\psi}_n}{dy} = -\frac{C^r_{1212}}{C_{1212}(y)}\frac{D_{yy}}{I_{yy}}\int_{-1/2}^{y}\frac{C^*_{1111}(\hat{y})}{C^r_{1111}}\bar{\psi}_{n-1}(\hat{y})\,d\hat{y}, \quad n = 1,\ldots,n_f \tag{63}$$

with $\bar{\psi}_0 = y$. Equation (63) is integrated making use of the interface conditions $[\![\bar{\psi}_n]\!] = 0$ at $y = y_i (i = 1,\ldots,N-1)$ and the (weighted) null mean value condition:

$$\int_{-1/2}^{1/2} C^*_{1111}\bar{\psi}_n\,dy = 0, \quad n = 1,\ldots,n_f \tag{64}$$

Equations (62) and (63) correspond to the exact definition of the cross-sectional warpings for the interior problem of a transversely indeformable multilayered beam subject to a transverse load varying according to a polynomial law. In particular, a number $n_f = \text{int}[(p+1)/2]$ of warping functions provides for the exact solution for a transverse load of order $p$. The coordinate function $\psi_n$ is given by a polynomial law of order $2n+1$ through each individual layer and presents discontinuous derivative at the layer interfaces, as is required to satisfy the shear stress continuity reported in eqns (2a).

## 6. A SIMPLE CASE: CONSTANT SHEAR RESULTANT

The simple case of a multilayered beam subject to a constant shear resultant $T$ is useful to understand the mechanical meaning of the additional axial and transverse displacements $\Delta u_1$ and $\Delta u_2$ derived in Section 4.

The 1-D solution given by SHDT is considered first, which yields the exact solution (under the assumption of transverse inextensibility) in the form of eqns (9) and (10), with one warping function only [$n_f = 1$ in eqns (62) and (63)]. Making use of boundary conditions $\xi = 0$, $\varphi = 0$ at $x = 0$ and $\varphi' = 0$ at $x = 1$, the transverse deflection, the average rotation and the warping amplitude are obtained in the form:

$$u_2^*(x) = L\eta(x) = \frac{T}{2C^r_{1111}I_{yy}}\left(\frac{L}{h}\right)^3 x^2\left(1-\frac{x}{3}\right) + \frac{T}{C^r_{1212}k}\frac{L}{h}x,$$

$$\varphi(x) = \frac{T}{C^r_{1111}I_{yy}}\frac{L^2}{h^2}x\left(1-\frac{x}{2}\right),$$

$$\chi(x) = -\frac{D_{y\psi}}{D_{\psi\psi}}\frac{T}{C^r_{1212}kh} = \text{const}, \tag{65}$$

where $k$ is the shear correction factor for the multilayered beam defined as:

$$k = \frac{D_{\chi\chi} D_{\psi\psi} - D_{\chi\psi}^2}{D_{\psi\psi}}. \tag{66}$$

Following the guidelines of Section 4, the following S admissible stress field:

$$\bar{\sigma}_{11} = C_{1111}^{*} \frac{Tx}{C'_{1111} I_{\chi\chi}} \frac{L}{h^2} y, \quad \bar{\sigma}_{12} = C_{1212}\left[1 - \frac{D_{\chi\psi}}{D_{\psi\psi}} \frac{d\psi}{dy}\right] \frac{T}{C'_{1212} kh}, \quad \bar{\sigma}_{22} = 0 \tag{67}$$

and refined displacement field:

$$u_1 = h[-\varphi y + \chi\psi(y)] - \frac{T}{C'_{1111} I_{\chi\chi}} \int_0^x n_x(\hat{y})\, d\hat{y},$$

$$u_2 = \frac{T}{2C'_{1111} I_{\chi\chi}} \left(\frac{L}{h}\right)^3 x^2 \left(1 - \frac{x}{3}\right) + \frac{T}{C'_{1212} k} \frac{L}{h} x + \frac{Tx}{C'_{1111} I_{\chi\chi}} \frac{L}{h} n_y(y) \tag{68}$$

are derived. Since the additional term in the axial displacement (68a) is constant along $x$, eqns (35) state that the K admissible stress field, which can be obtained from eqns (68), coincides with eqns (67). Hence, eqns (67) and (68) represent the exact 2-D solution of the problem. Note that the SHDT 1-D solution of eqn (65) differs from the exact displacement field (68) for terms related to the Poisson coefficient only. It can be verified that for homogeneous and three-layered isotropic beams this solution reduces to those obtained by Timoshenko and Goodier (1970), and Sierakowski and Ebcioglu (1970), respectively, by making use of Airy stress function.

For a constant shear resultant, the exact 2-D solution can also be obtained starting from the 1-D displacement field given by FSDT or even by CLT. For instance, starting from eqns (38) and CLT lateral deflection:

$$u_2^* = L\eta(x) = \frac{T}{2C'_{1111} I_{\chi\chi}} \left(\frac{L}{h}\right)^3 x^2 \left(1 - \frac{x}{3}\right) + \varphi_0 Lx, \tag{69}$$

where $\varphi_0$ is the rotation at $x = 0$, the following 2-D displacement field is derived:

$$u_1 = -\frac{T}{C'_{1111} I_{\chi\chi}} \left(\frac{L}{h}\right)^2 x\left(1 - \frac{x}{2}\right) y + \frac{T}{C'_{1111} I_{\chi\chi}} \int_0^y S_{1212} \bar{s}_{12,}(\hat{y})\, d\hat{y}$$

$$- \frac{T}{C'_{1111} I_{\chi\chi}} \int_0^x n_x(\hat{y})\, d\hat{y} - \varphi_0 hy,$$

$$u_2 = \frac{T}{2C'_{1111} I_{\chi\chi}} \left(\frac{L}{h}\right)^3 x^2 \left(1 - \frac{x}{3}\right) + \frac{Tx}{C'_{1111} I_{\chi\chi}} \frac{L}{h} n_y(y) + \varphi_0 Lx \tag{70}$$

which coincides with eqns (68), apart from a rigid body rotation given by:

$$\varphi_0 = \frac{T}{C'_{1212} kh}. \tag{71a}$$

In particular, the second term at the right-hand side of eqn (70a) represents, to the accuracy of the rigid body rotation $\varphi_0$, the cross-sectional warping due to shear strains (see

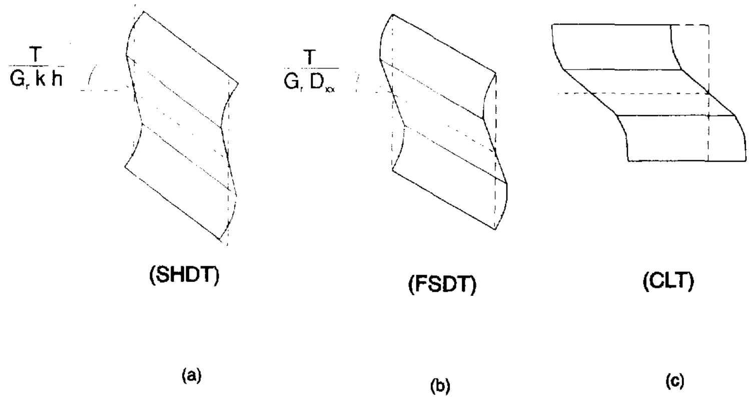



Fig. 2. Cross-sectional warping due to shear strains for a typical three-layered beam with a soft internal core.

Fig. 2). The same displacement field can be obtained starting from FSDT. In this case, the rigid body rotation turns out to be (see Fig. 2):

$$\varphi_0 = \frac{T}{C'_{1212}h}\left(\frac{1}{k} - \frac{1}{D_\infty}\right). \tag{71b}$$

## 7. NUMERICAL EXAMPLES

In this section, the accuracy of classical and higher order 1-D models is estimated with reference to simply-supported multilayered beams under sinusoidal transverse load:

$$p(x) = p_n \sin \alpha_n x, \tag{72}$$

where $\alpha_n = n\pi$. For higher order 1-D models, the solution can be derived by setting:

$$\eta(x) = \eta_n \sin \alpha_n x, \quad \varphi(x) = \varphi_n \cos \alpha_n x, \quad \chi(x) = \chi_n \cos \alpha_n x \tag{73}$$

so reducing eqns (15) to an algebraic system for the unknown coefficients $\eta_n$, $\varphi_n$, $\chi_n$. In this case, the solution wavelengths defined in eqns (46) and (50a) are $l_1 = l_2 = l = L/n\pi$ and, correspondingly, for all the 1-D models $\hat{C}^{\mathrm{I}} = C^{\mathrm{I}}/n\pi$ and $\hat{C}^{\mathrm{II}} = C^{\mathrm{II}}/(n\pi)^2$.

In the numerical examples, $n = 1$ is considered and three different sets of elastic coefficients are adopted, corresponding to two isotropic materials (denoted by I1 and I2) and an orthotropic (O1) material:

$$\text{(I1)} \quad E_1 = E_2 = 200\,\text{GPa}, \qquad G_{12} = 77\,\text{GPa}, \quad \nu_{12} = 0.3;$$

$$\text{(I2)} \quad E_1 = E_2 = 10\,\text{GPa}, \qquad G_{12} = 4\,\text{GPa}, \quad \nu_{12} = 0.25;$$

$$\text{(O1)} \quad E_1 = 200\,\text{GPa}, \quad E_2 = 12\,\text{GPa}, \quad G_{12} = 8\,\text{GPa}, \quad \nu_{12} = 0.3.$$

Figures 3–6 show the actual relative errors $e^{\mathrm{I}}$ and $e^{\mathrm{II}}$ [with respect to the exact 2-D solution by Pagano (1969)] as a function of the beam height-to-length ratio, for a single-layer isotropic (I1) beam, a single-layer orthotropic (O1) beam, a three-layered isotropic (I1–I2–I1) beam and a three-layered orthotropic (O1–I2–O1) beam. The errors have been computed for CLT, FSDT, LHDT and SHDT. As for the two higher order theories, the number of terms adopted for the axial displacement is reported in parentheses. For each theory, dashed and solid lines denote I-order and II-order error bounds, respectively. All the figures confirm the asymptotic behavior for $L/h \to \infty$ predicted in Sections 4 and 5, that is of $O(h/L)$ for the I-order error bounds and $O(h^2/L^2)$ for the II-order error bounds. The I-order and II-order accuracy coefficient $\hat{C}^{\mathrm{I}}$ and $\hat{C}^{\mathrm{II}}$ for the four beams considered are reported in Tables 2 and 3. Since the I-order error bounds (dashed lines) are dominated by

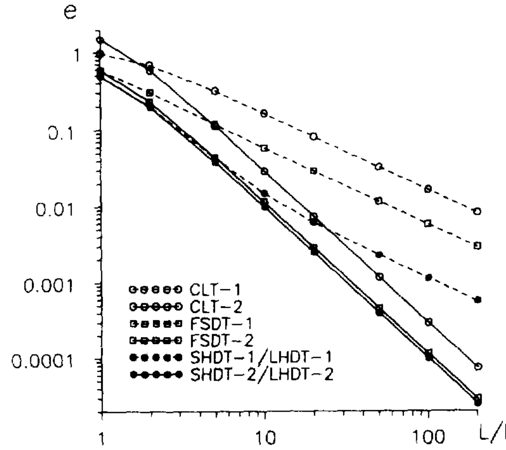



Fig. 3. Isotropic single-layer beam under sinusoidal transverse loading: actual relative error bounds versus length-to-height ratio for classical lamination theory (CLT), first order shear deformation theory (FSDT) and for Lo *et al.* (LHDT) and Savoia *et al.* (SHDT) higher order models. 1 and 2 stand for I- and II-order error bound, respectively.

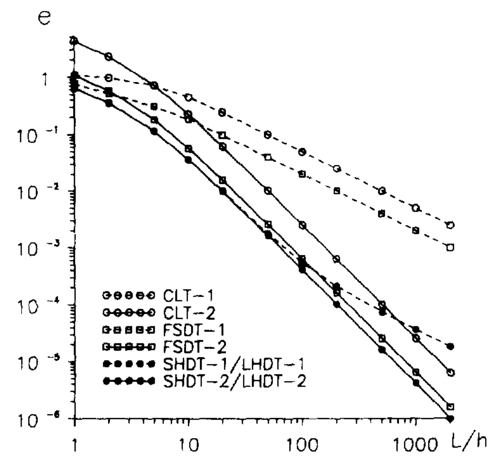
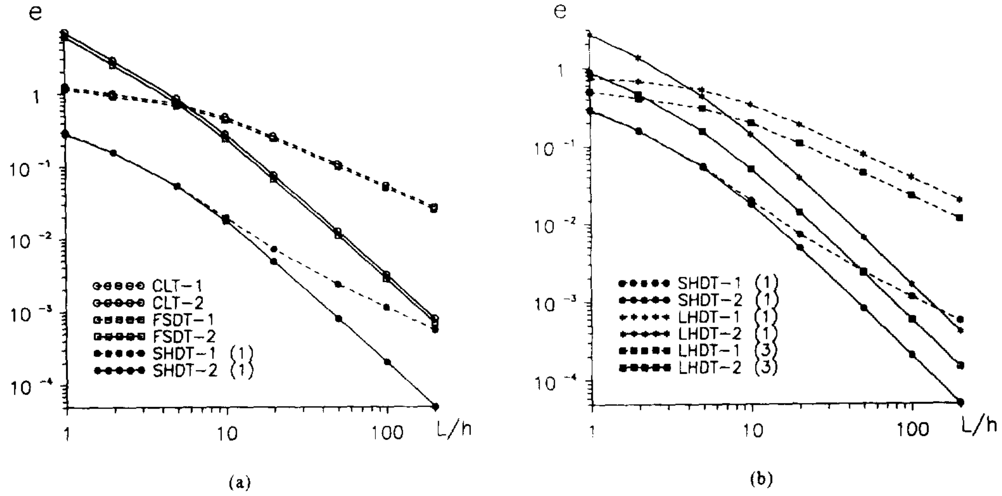
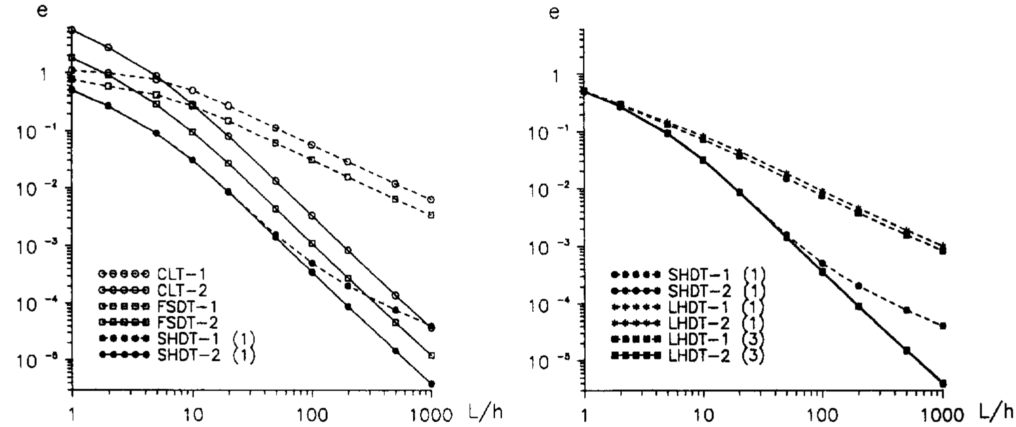



Fig. 4. Orthotropic single-layer beam: actual relative error bound versus length-to-height ratio for CLT, FSDT, LHDT and SHDT. 1 and 2 stand for I- and II-order error bound, respectively.




(a) (b)

Fig. 5. (a,b) Isotropic three-layered beam: actual relative error bound versus length-to-height ratio for CLT, FSDT, LHDT and SHDT. 1 and 2 stand for I- and II-order error bound, respectively. The number of coordinate functions adopted for axial displacement is reported in parentheses.




(a) (b)

Fig. 6. (a,b) Orthotropic three-layered beam: actual relative error versus length-to-height ratio for CLT, FSDT, LHDT and SHDT. 1 and 2 stand for I- and II-order error bound, respectively. The number of coordinate functions adopted for axial displacement is reported in parentheses.

Table 2. I-order accuracy coefficient $\hat{C}^{I}$ for single-layer isotropic (I1) beam, single-layer orthotropic (O1) beam, three-layered isotropic (I1–I2–I1) beam and three-layered orthotropic (O1–I2–O1) beam

| $\hat{e}^{I} \approx \hat{C}^{I}(h/l)$ 1-D model | (I1) | (O1) | (I1–I2–I1) | (O1–I2–O1) |
|---|---|---|---|---|
| CLT | 0.5255 | 1.586 | 1.750 | 1.841 |
| FSDT | 0.1860 | 0.6379 | 1.619 | 0.9973 |
| Lo *et al* (1) | 0.0360 | 0.01162 | 1.235 | 1.171 |
| Lo *et al* (2) | 0.0360 | 0.01162 | 0.7509 | 0.2869 |
| Lo *et al* (3) | 0.0360 | 0.01162 | 0.7130 | 0.2405 |
| Savoia *et al* (1) | 0.0360 | 0.01162 | 0.03559 | 0.01165 |

Table 3. II-order accuracy coefficient $\hat{C}^{II}$ for single-layer isotropic (I1) beam, single-layer orthotropic (O1) beam, three-layered isotropic (I1–I2–I1) beam and three-layered orthotropic (O1–I2–O1) beam

| $\hat{e}^{II} \approx \hat{C}^{II}(h/l)^2$ 1-D model | (I1) | (O1) | (I1–I2–I1) | (O1–I2–O1) |
|---|---|---|---|---|
| CLT | 0.2988 | 2.561 | 3.239 | 3.428 |
| FSDT | 0.1168 | 0.6501 | 2.827 | 1.132 |
| Lo *et al* (1) | 0.1010 | 0.4108 | 1.638 | 0.3729 |
| Lo *et al* (2) | 0.1010 | 0.4108 | 0.6381 | 0.3720 |
| Lo *et al* (3) | 0.1010 | 0.4108 | 0.5856 | 0.3651 |
| Savoia *et al* (1) | 0.1010 | 0.4108 | 0.2048 | 0.3584 |

the difference of shear stresses [see eqn (32)], the higher order theories represent a notable improvement over both CLT and FSDT. For instance, for the O1-beam, CLT and FSDT yield coefficient $\hat{C}^{I}$ even 136.5 and 54.9 times greater than higher order theories (see Table 2).

It is worth noting that, for single-layer beams, the coordinate functions adopted by the two higher order models are the same, so that they give exactly the same results.

Moreover, the asymptotic behavior is independent of the number of terms adopted. In fact, for $L/h \to \infty$ the load wavelength tends to infinity [see eqn (72)], and the corresponding solution requires only the (first) cubic Legendre's polynomial for the axial displacement. The behavior of two higher order models is completely different for laminated beams. For instance, the error related to SHDT is substantially the same for I1-beam and I1–I2–I1-beam as well as for O1-beam and O1–I2–O1–beam (see Table 2), and it is still independent of the number of terms adopted. In fact, the first coordinate function $\psi_1(y)$ defined in eqns (62) and (63) provides for the closed form solution for a uniform transverse load for any beam lamination. On the contrary, LHDT does not satisfy the shear stress continuity at the interfaces and yields an error more than 20 times greater than SHDT for both laminated beams (see Table 2). Moreover, CLT gives an error 49.2 and 158 times greater than SHDT for the isotropic and the orthotropic laminated beams, respectively.

Finally, Figs 3–6 clearly show that the asymptotic behavior of the error bounds predicted by eqns (31) and (36) is valid for $L/h \to \infty$, whereas it can lead to erroneous conclusions when beams of finite length are considered. For instance, Figs 4 and 5 show that, for laminated beams, the I-order error bound for SHDT is narrower than the II-order error bound for FSDT if $L/h < 200$, and even narrower than the II-order error bound for LHDT(3) if $L/h < 50$. Moreover, for classical theories and LHDT, the II-order error bound can be even wider than the I-order error bound if thick ($L/h < 10$) beams are considered. This is not the case of SHDT, where the improved K-admissible solution represents an improvement over the lower accuracy solution for the whole range of $L/h$.

In Figs 7 and 8 the shear stress distributions at $x = 0$ given by 1-D models for a thick ($L/h = 4$) orthotropic O1–I2–O1-beam are compared with the exact solution. It is worth noting that SHDT yields very accurate shear stresses (Fig. 7a), whereas this is not the case of LHDT (Fig. 7b), due to the jumps at the layer interfaces which cannot be reduced by increasing the number of coordinate functions. Figure 8 shows the stress distributions derived from FSDT, i.e. the basic 1-D solution, the S admissible and the lower accuracy K admissible solutions. As announced in Section 4, the contribution to shear stresses included in the K admissible solution does not represent a significant improvement over 1-D solution. On the contrary, the S admissible solution is a good approximation of the exact solution. This circumstance represents the main motivation of Danielson' technique of deriving the improved K admissible stress field by making the related shear stress equal to the S admissible shear stress.

For the three-layered isotropic (I1–I2–I1) and orthotropic (O1–I2–O1) beams, the coefficient $\hat{C}^{\mathrm{I}}$ is reported in Fig. 9a,b as a function of the face thickness ratio $\delta_f = 2d/h$. For the isotropic beam (Fig. 9a), SHDT yields coefficients $\hat{C}^{\mathrm{I}}$ which are substantially independent of $\delta_f$ ($\hat{C}^{\mathrm{I}} = 0.0306$ for $\delta_f = 0$ and $= 0.0360$ for $\delta_f = 1$). On the contrary, the accuracy of both classical theories and LHDT strongly depends on the relative thicknesses of faces and core. For instance, $\hat{C}^{\mathrm{I}}$ for FSDT is equal to 0.1851 for homogeneous beams ($\delta_f = 0$ and $\delta_f = 1$) and rises up to 1.6536 for $\delta_f = 0.5$. In fact, the assumption of linear variation of axial displacement over the beam height breaks down for laminated beams. As for LHDT, Fig. 9a shows that for $\delta_f > 0.7$ the addition of more terms in the displacement expansion does not represent a significant improvement over FSDT, whereas it does for $\delta_f < 0.5$. For the orthotropic beam (Fig. 9b) similar conclusions can be drawn. In this case, the maximum error occurs for $\delta_f = 0.6$ for classical theories and for $\delta_f = 0.8$ for LHDT(3). On the contrary, $\hat{C}^{\mathrm{I}}$ given by SHDT decreases monotonically from $\delta_f = 0$ ($\hat{C}^{\mathrm{I}} = 0.0306$) to $\delta_f = 0.6$ ($\hat{C}^{\mathrm{I}} = 0.0115$) and is almost constant up to $\delta_f = 1$ ($\hat{C}^{\mathrm{I}} = 0.0116$).

For the same two laminated beams, coefficients $\hat{C}^{\mathrm{II}}$ are reported in Fig. 10a,b. Note that the accuracy of LHDT significantly depends on the beam lamination: for the isotropic beam with $\delta_f < 0.2$ the accuracy of LHDT(3) and SHDT(1) are comparable whereas, for $\delta_f = 0.8$, $\hat{C}^{\mathrm{II}}$ for LHDT(3) is 13.7 times greater than for SHDT(1).

Finally, coefficients $\hat{C}^{\mathrm{I}}$ and $\hat{C}^{\mathrm{II}}$ for laminated isotropic beams with $\delta_f = 0.4$ are reported in Fig. 11a and b as a function of the Young modulus ratio $r_f = E^{(1)}/E^{(2)}$ of external layers and internal core. For the core, $E^{(2)}$ and $G^{(2)}$ correspond to I2-material whereas, for the external faces, Young modulus $E^{(1)}$ ranging from $0.02\ E^{(2)}$ to $20\ E^{(2)}$ and shear modulus $G^{(1)} = E^{(1)}/2.5$ are considered. As for SHDT, the I-order accuracy coefficient (Fig. 11a)

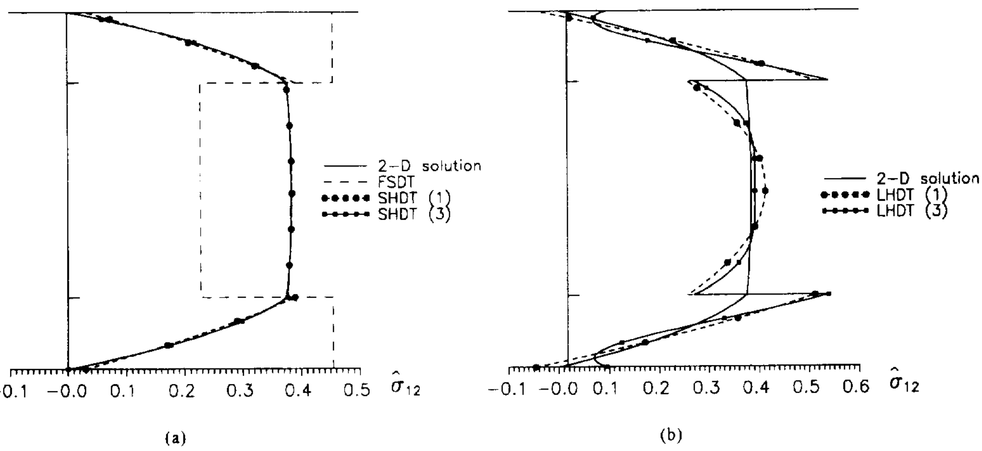



Fig. 7. (a,b) Shear stress distributions $\hat{\sigma}_{12} = \sigma_{12}(0, y)h/p_1L$ for the orthotropic (O1–I2–O1) beam. The results given by FSDT, LHDT and SHDT are compared with the exact 2-D solution by Pagano (1969). The number of coordinate functions adopted for axial displacement is reported in parentheses.

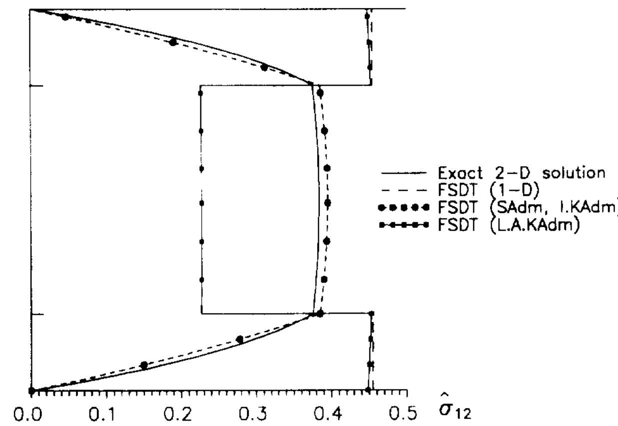
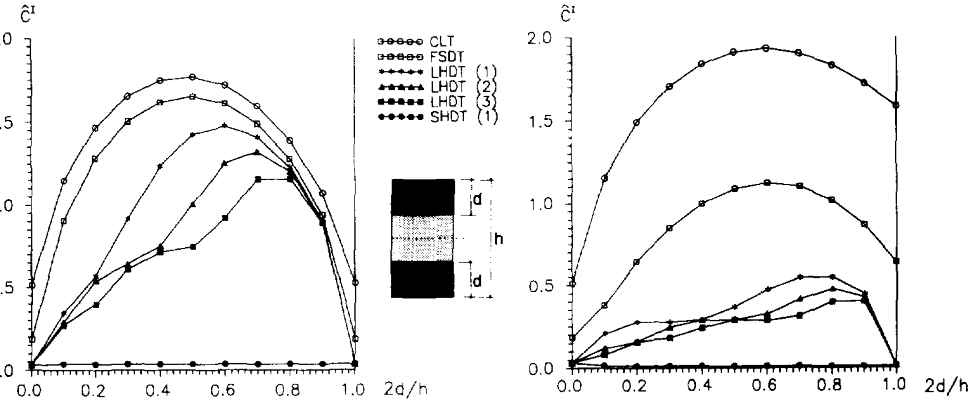



Fig. 8. Shear stress distributions $\hat{\sigma}_{12} = \sigma_{12}(0, y)h/p_1L$ for the orthotropic three-layered beam obtained by refining the results given by FSDT: 1-D solution, S admissible and improved K admissible solution, lower accuracy K admissible solution.

weakly depends on $r_E$, varying from 0.0197 for $r_E = 0.02$ to 0.0330 for $r_E = 20$. On the contrary, the accuracy of all the other theories varies significantly if external faces softer or stiffer than the core are considered. For instance, $\hat{C}^{\text{I}}$ for LHDT(3) is always less than 0.0675 for $r_E \leqslant 1$, but it strongly increases when stiff faces are considered, raising up to 0.7168 for $r_E = 20$. Finally, Fig. 11b confirms that the improved K admissible solution represents a significant improvement for classical theories and LHDT. In fact, when $r_E < 2$ ($r_E < 10$), $\hat{C}^{\text{II}}$ given by FSDT (LHDT) are comparable with that given by SHDT.

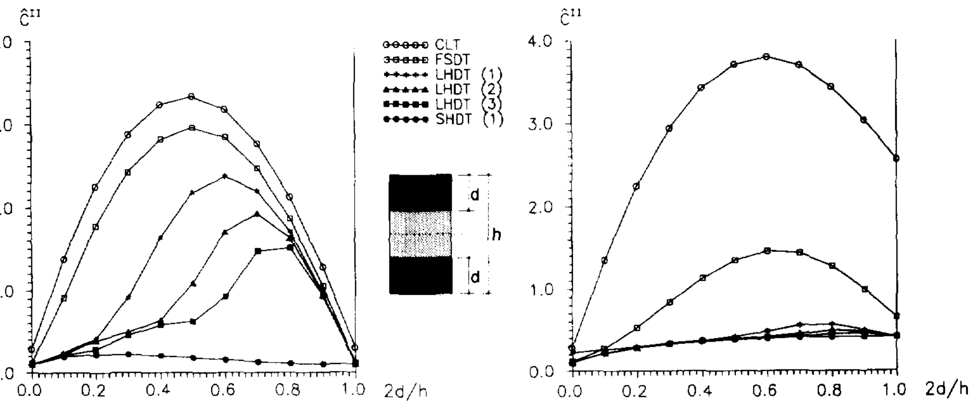
## CONCLUSIONS

The Prager–Synge hypercircle method has been used to derive error bounds for classical and higher order laminated beam theories. A statically admissible and a lower accuracy kinematically admissible stress field are derived, whose relative mean-square error is $O(h/l)$ for $h/l \to 0$. Then, a generalization of Danielson's technique to orthotropic multilayered beams is given, in order to obtain an improved kinematically admissible stress field and a corresponding asymptotic form for the relative error of $O(h^2/l^2)$. Two coefficients $\hat{C}^{\text{I}}$ and $\hat{C}^{\text{II}}$, named I- and II-order accuracy coefficients, are defined and employed to estimate the accuracy of stress fields derived from 1-D models. These coefficients are given in explicit form for CLT, FSDT and higher order theories by Lo *et al.* (1977) (LHDT) and Savoia *et al.*



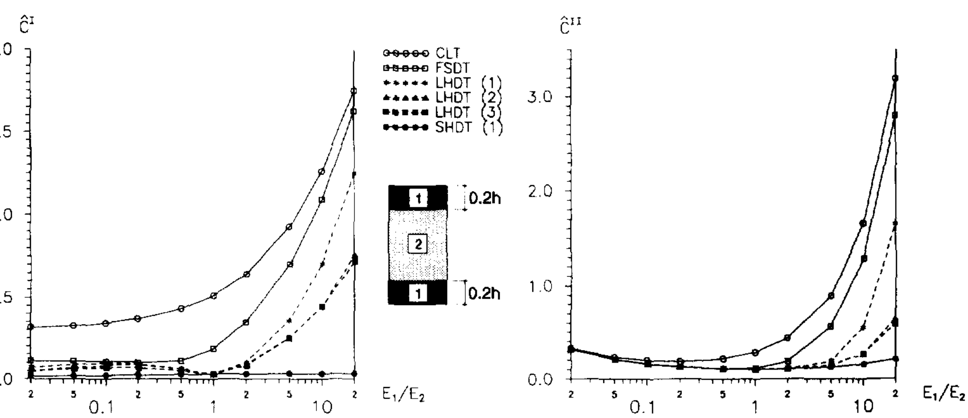

(a) (b)

Fig. 9. (a) Isotropic and (b) orthotropic three-layered beams: I-order accuracy coefficient $\hat{C}^{\mathrm{I}}$ versus face thickness ratio $\delta_f = 2d/h$ for CLT, FSDT, LHDT and SHDT.




(a) (b)

Fig. 10. (a) Isotropic and (b) orthotropic three-layered beams: II-order accuracy coefficient $\hat{C}^{\mathrm{II}}$ versus face thickness ratio $\delta_f = 2d/h$ for CLT, FSDT, LHDT and SHDT.




(a) (b)

Fig. 11. Isotropic three-layered beam: (a) I-order accuracy coefficient $\hat{C}^{\mathrm{I}}$ and (b) II-order accuracy coefficient $\hat{C}^{\mathrm{II}}$ versus Young modulus ratio of layers $r_E = E^{(1)}/E^{(2)}$ for CLT, FSDT, LHDT and SHDT.

(1993) (SHDT), and are quantitatively computed for multilayered beams under sinusoidal transverse loading.

The examples presented show that, unlike the single-layer models based on $C^\infty$-continuous coordinate functions (CLT, FSDT, LHDT), the accuracy of SHDT is substantially independent of the beam lamination and degree of orthotropy of the individual layers. Moreover, the I-order error bound derived from SHDT can be even 150 and 20 times narrower than those given by CLT and LHDT, due to the very accurate representation of shear stresses over the beam height. On the contrary, the improved axial displacement represents a good improvement for LHDT, so that the corresponding error bound is only slightly wider than for SHDT.

In the numerical examples, it has been shown that even for isotropic laminated beams, the error bound of $O(h^2/L^2)$ for FSDT and LHDT can be much wider than the error bound of $O(h/L)$ for SHDT if $L/h < 200$ and $L/h < 50$, respectively. This circumstance represents the main motivation to develop refined 1-D higher order beam theories, yielding *ab initio* accurate stress distributions.

The proposed error bounds apply to displacement-based models based on the kinematic assumption of inextensibility in the transverse direction. This assumption is particularly appropriate for orthotropic beams. For instance, Sayir (1980) performed an asymptotic expansion of 2-D elasticity equations for homogeneous strongly orthotropic materials $[E_1/G_{12} = O(H^2/L^2)]$, obtaining a set of displacement equilibrium equations where the transverse displacement is constant through the beam height. Moreover, in Savoia and Tullini (1994) it has been proved that both the interior and boundary exact solutions reduce to those given by the 1-D theory when $G_{12}/E_1 \to 0$.

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## REFERENCES

Andreev, A. N. and Nemirovskii, Y. V. (1977). On the theory of multilayered elastic anisotropic shells. *Izv. Akad. Nauk. SSSR, Mekh. Tverd. Tela* **12**, 87–96 (Engl. transl. *Mech. Solids* **12**, 73–81).

Chepiga, V. E. (1977). On constructing a theory of multilayered anisotropic shells with prescribed arbitrary accuracy of order $h^N$. *Izv. Akad. Nauk. SSSR, Mekh. Tverd. Tela* **12**, 113–120 (Engl. transl. *Mech. Solids* **12**, 98–103).

Choi, I. and Horgan, C. O. (1977). Saint-Venant's principle and end effects in anisotropic elasticity. *J. Appl. Mech. ASME* **44**, 424–430.

Cicala, P. (1962). Consistent approximations in shell theory. *J. Engng Mech. Dir., Proc. Am. Soc. Civ. Eng.* **88**, 45–74.

Danielson, D. A. (1971). Improved error estimates in the linear theory of thin elastic shells. *Proc. Kon. Ned. Akad.* **B74**, 294–300.

Duva, J. M. and Simmonds, J. G. (1990). Elementary, static beam theory is as accurate as you please. *J. Appl. Mech. ASME* **57**, 134–137.

Duva, J. M. and Simmonds, J. G. (1991). The usefulness of elementary beam theory for the linear vibrations of layered, orthotropic elastic beams and corrections due to two-dimensional end effects. *J. Appl. Mech. ASME* **58**, 175–180.

Duva, J. M. and Simmonds, J. G. (1992). The influence of two-dimensional end effects on the natural frequencies of cantilever beams weak in shear. *J. Appl. Mech. ASME* **59**, 230–232.

Koiter, W. T. (1970). On the foundation of the linear theory of thin elastic shells. *Proc. Kon. Ned. Akad. Wet.* **B73**, 169–195.

Ladevèze, P. (1976). Justification de la théorie linéaire des coques élastiques. *J. Mécanique* **15**, 813–856.

Ladevèze, P. (1980). On the validity of linear shell theories. *Theory of Shells* (Edited by W. T. Koiter and G. K. Mikhailov), pp. 367–391. North-Holland, Amsterdam.

Ladevèze, P. and Pecastaings, F. (1988). The optimal version of Reissner's theory. *J. Appl. Mech. ASME* **55**, 413–418.

Levinson, M. (1981). A new rectangular beam theory. *J. Sound Vibr.* **74**, 81–87.

Lewinski, T. (1987). On refined plate models based on kinematical assumptions. *Ing. Archiv.* **57**, 133–146.

Librescu, L. (1967). On the theory of anisotropic elastic shells and plates. *Int. J. Solids Structures* **3**, 53–68.

Lo, K. H., Christensen, R. M. and Wu, E. M. (1977). A higher-order theory of plate deformation: II) Laminated plates. *J. Appl. Mech. ASME* **44**, 669–676.

Murakami, H. (1986). Laminated composite plate theory with improved in-plane response. *J. Appl. Mech. ASME* **53**, 661–666.

Nordgren, R. P. (1971). A bound on the error in plate theory. *Quart. Appl. Math.* **29**, 587–595.

Nordgren, R. P. (1972). A bound on the error in Reissner's theory of plates. *Quart. Appl. Math.* **30**, 551–556.

Pagano, N. J. (1969). Exact solutions for composite laminates in cylindrical bending. *J. Compos. Mater.* **3,** 398–411.

Podio Guidugli, P. (1989). An exact derivation of the thin plate equation. *J. Elasticity* **22,** 121–133.

Prager, W. and Synge, J. L. (1947). Approximation in elasticity based on the concept of function space. *Quart. Appl. Math.* **5,** 241–269.

Rasskazov, A. O., Sokolovskaya, I. I. and Shul'ga, N. A. (1983). Comparative analysis of several shear models in problems of equilibrium and vibrations for multilayered plates. *Priklad. Mekh.* **19,** 84–90 (Engl. transl. *Sov. Appl. Mech.* **19,** 633–638).

Reddy, J. N. (1987). A generalization of two-dimensional theories of laminated composite plates. *Commun. Appl. Num. Meth.* **3,** 173–180.

Reissner, E. (1944). On the theory of bending of elastic plates. *J. Math. Phys.* **23,** 184–191.

Reissner, E. (1963). On the derivation of the theory of thin elastic shells. *J. Math. Phys.* **42,** 263–277.

Reissner, E. (1985). Reflections on the theory of elastic plates. *Appl. Mech. Rev.* **38,** 1453–1464.

Reissner, E. and Stavsky, Y. (1961). Bending and stretching of certain types of heterogeneous aelotropic elastic plates. *J. Appl. Mech. ASME* **28,** 402–408.

Rychter, Z. (1987a). An engineering theory for beam bending. *Ing. Arch.* **58,** 25–34.

Rychter, Z. (1987b). On the accuracy of a beam theory. *Mech. Res. Commun.* **14,** 99–105.

Rychter, Z. (1987c). A sixth-order plate theory. Derivation and error estimates. *J. Appl. Mech. ASME* **54,** 275–279.

Rychter, Z. (1987d). A refined bound on the error of Reissner's plate theory. *Ing. Archiv* **57,** 277–286.

Rychter, Z. (1987e). Error estimates for a sixth-order theory of plate bending. *J. Engng Math.* **21,** 263–270.

Rychter, Z. (1988a). An improved error estimate for Reissner's plate theory. *Int. J. Solids Structures* **24,** 537–544.

Rychter, Z. (1988b). A simple and accurate beam theory. *Acta Mechan.* **75,** 57–62.

Rychter, Z. (1992). A family of accurate plate theories. *Mech. Res. Commun.* **19,** 505–510.

Rychter, Z. (1993). Generalized displacements and the accuracy of classical plate theory. *Int. J. Solids Structures* **30,** 129–136.

Savoia, M. and Reddy, J. N. (1992). A variational approach to three-dimensional elasticity solutions of laminated composite plates. *J. Appl. Mech. ASME* **59,** S166–S175.

Savoia, M. and Tullini, N. (1994). A beam theory for strongly orthotropic materials. submitted.

Savoia, M., Laudiero, F. and Tralli, A. (1993). A refined theory for laminated beams – Part I: A new high order approach. *Meccanica* **28,** 39–51.

Savoia, M., Laudiero, F. and Tralli, A. (1994). A two-dimensional theory for the analysis of laminated plates. *Computat. Mech.* **14,** 38–51.

Sayir, M. (1980). Flexural vibrations of strongly anisotropic beams. *Ing. Arch.* **49,** 309–330.

Sierakowski, R. L. and Ebcioglu, I. K. (1970). On interlaminar shear stresses in composites. *J. Compos. Mater.*, **4,** 144–149.

Simmonds, J. G. (1971). An improved estimate for the error in the classical linear theory of plate bending. *Quart. Appl. Math.* **29,** 439–447.

Timoshenko, S. P. and Goodier, J. N. (1970). *Theory of Elasticity*, 3rd Ed. McGraw Hill, New York.

Truesdell, C. and Noll, W. (1965). The non-linear field theories of mechanics. In *Encyclopedia of Physics* (Edited by S. Flugge), Vol. III/3. Springer, Berlin.

Van Keulen, F. (1991). A bound on the error in classical theory of laminated plates. *Int. J. Engng Sci.* **29,** 869–882.

Whitney, J. M. and Pagano, N. J. (1970). Shear deformation in heterogeneous anisotropic plates. *J. Appl. Mech. ASME* **37,** 1031–1036.

## APPENDIX

Equations (15) are rewritten as follows:

$$(\eta' - \varphi) + \frac{\mathbf{D}^{t}_{\chi\varphi}}{D_{\varphi\varphi}}\boldsymbol{\chi} = \frac{\mathbf{D}_{\chi\varphi}}{C'_{1212}D_{\varphi\varphi}h}T(x),$$

$$C'_{1111}\left(\frac{h}{L}\right)^{2} I_{\varphi\varphi}\varphi'' = \frac{T(x)}{h},$$

$$\frac{C'_{1111}}{C'_{1212}}\left(\frac{h}{L}\right)^{2}\mathbf{I}_{\psi\psi}\boldsymbol{\chi}'' - \mathbf{A}\boldsymbol{\chi} = \frac{\mathbf{D}_{\chi\varphi}}{C'_{1212}D_{\varphi\varphi}h}T(x), \tag{A1}$$

where $T(x) = -L\int_{0}^{x} p(\hat{x})\,\mathrm{d}\hat{x} + T_{0}$ is the shear resultant and matrix $\mathbf{A}$ is defined as:

$$\mathbf{A} = \frac{\mathbf{D}_{\psi\psi}D_{\varphi\varphi} - \mathbf{D}_{\chi\varphi}\mathbf{D}^{t}_{\chi\varphi}}{D_{\varphi\varphi}}. \tag{A2}$$

From eqns (A1) the average shear deformation $\eta' - \varphi$ and vector $\boldsymbol{\chi}$ can be expressed in terms of second derivatives of $\varphi$ and $\boldsymbol{\chi}$:

$$\eta' - \varphi = \left(\frac{h}{L}\right)^{2}[\mathbf{g}^{t}_{\psi}\boldsymbol{\chi}'' + g_{\varphi}\varphi''], \quad \boldsymbol{\chi} = \left(\frac{h}{L}\right)^{2}[\mathbf{E}_{\psi}\boldsymbol{\chi}'' + \mathbf{e}_{\varphi}\varphi'']. \tag{A3}$$

where

$$\mathbf{E}_\psi = \frac{C'_{1111}}{C'_{1212}}\mathbf{A}^{-1}\mathbf{I}_{\psi\psi}, \quad \mathbf{e}_y = \frac{C'_{1111}}{C'_{1212}}\mathbf{A}^{-1}\frac{\mathbf{D}_{y\psi}}{D_{yy}}I_{yy}, \tag{A4}$$

$$\mathbf{g}_\psi^{\mathrm{T}} = -\frac{C'_{1111}}{C'_{1212}}\frac{\mathbf{D}_{y\psi}^{\mathrm{T}}}{D_{yy}}\mathbf{A}^{-1}\mathbf{I}_{\psi\psi}, \quad g_y = -\frac{C'_{1111}}{C'_{1212}}\left(\frac{\mathbf{D}_{y\psi}^{\mathrm{T}}\mathbf{A}^{-1}\mathbf{D}_{y\psi}}{D_{yy}^2}+1\right)I_{yy}.$$

Then, making use of eqns (14) and (A3) the shear stress can be written in the form of eqn (51). Finally, substituting eqns (A3) in eqns (A1a,b), straightforward algebra yields:

$$\varphi' = c_1\frac{L}{h^3}M, \quad \varphi'' = c_y\frac{L^2}{h^3}M^{[1]}, \quad \boldsymbol{\chi}'' = \mathbf{c}_\psi\frac{L^2}{h^3}M^{[1]}, \tag{A5}$$

where $c_y$ and $\mathbf{c}_\psi$ are sets of coefficients which depend on the beam lamination and the coordinate functions adopted.